A martingale approach to the question of fiscal stimulus

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Abstract

Democrats in the United States argue that government spending can be used to "grease the wheels" of the economy to create wealth and to increase employment; Republicans contend that government spending is wasteful and discourages investment, thereby increasing *unemployment*. These arguments cannot both be correct, but both arguments seem meritorious. Faced with this paradox, one might hope that a rigorous mathematical approach might help determine the truth. We address this economic question of fiscal stimulus as a new optimal control problem generalizing the model of Dutta & Radner (1999). We find that there exists an optimal strategy and provide rigorous verification proof for the optimality. Further, we prove a few interesting mathematical properties of our solution, providing deeper insight into this important politico-economic debate and illustrating how the fiscal stimulus from the government may affect the profit-taking behavior of firms in the private sector.

Preprint submitted to SPA Special Issue: "In Memoriam: Larry Shepp" *April 28, 2019*

1. Introduction

The purpose of the present manuscript is to mathematically model optimal fiscal policy with the hopes of contributing to the ongoing debate between Democrats and Republicans on what measures should be taken to stimulate the economy. Although our original motivation for the present manuscript was to examine the question of taxation (should taxes be raised or lowered to reduce unemployment?), we instead succeed in formulating a rigorous mathematical model for government spending; taxes are still implicitly considered, but the control variable is how much – if at all – should the government inject into the private sector to improve the economic state variable wealth[1]. We can and do reach a firm conclusion with a reasonable model which turns out to favor the *democratic* side by showing that if the government injects money into the private sector, it will increase wealth enough, at least on average, to cover the expenditures which can then be recovered through taxation.

This would appear to settle the issue, at least for a mathematician, that truth lies on the democratic side. However, there is one important subtlety: if a company is free to take its profits in its own way so as to maximize its time discounted total dividends, then the conclusion reached above becomes false in the sense that there are some profitable companies where, if the company can set its own rule for taking dividends, the additional profit may *not* be enough to provide the old profits, or the company may even fail to pay the expected discounted loan cost. Thus, in order to obtain the advantage promised by the democratic side, the government would need to become intimately involved in the companies' internal decision makings. This weakens the conclusion in favor of the democrats if the greater society is not in favor of taking the freedom away from companies to use their own plan for profit taking.

The strategy needed to obtain sufficient wealth to pay for the mean discounted loans necessarily involves government control of how and when the company takes its profits. The answer to the question of whether or not mathematics can settle the debate of the effectiveness of fiscal stimulus then comes down to deciding whether society is willing to give up freedom of the company to set its profit-taking the way it wants and to accept some government interference. This decision is not included in our mathematical model, and it would appear not to be easily included in any mathematical model. Indeed,

[1]Fiscal policy refers to the budgetary decisions that the Government makes regarding revenue (via taxation) and expenditures (via spending). Politics plays as much if not more of a role than economics in fiscal policy, as evidenced by the seemingly endless quarreling between Democrats and Republicans. We summarize both sides' arguments here in the introduction, but also note that there is a rich economics literature on the topic with highly regarded economists justifying their respective sides. The contribution of the present work is to provide a rigorous mathematical framework using martingale theory to re-examine the issue.

this decision does not belong to mathematics.

The paper is organized as follows. In Section 2 we review the seminal work of Dutta & Radner (1999), which can be viewed as a baseline model where the government does not offer any loans to companies. Other related works are also briefly reviewed in Section 2.2. In Section 3 we propose our new model where the expected profit rate of a company can be increased by borrowing money from the government. We shall rigorously prove in Section 4 that the government's providing of loans to a company produces additional *net* profit.[2] Our model can also be used to mathematically analyze the behavior of "greedy" companies which do not care about repaying the government loans. Some theoretical results concerning such greedy behavior are provided in Section 4 and numerical studies are performed in Section 5. Rigorous verification arguments are give in Section 6.

2. Background

The purpose of the present section is to provide the requisite background for our new model in Section 3.

2.1. The model of Dutta & Radner (1999)

We follow Dutta & Radner (1999) and suppose that a company can be characterized by 4 parameters, (x, μ, σ, r), the present worth, x, the profit rate, μ, the riskiness, σ, and the prevailing interest rate, r. It is presumed that the objective of the firm is to maximize discounted total profit, and also that the company chooses its policy *optimally.*

Dutta & Radner (1999) formulated the following model for the dynamics of the wealth of the company:

$$\begin{aligned} dX(t) &= \mu dt + \sigma dW(t) - dZ(t),\ t \geq 0; \\ X(0-) &= x,\ W(0) = 0,\ Z(0-) = 0, \end{aligned} \tag{1}$$

[2]This would tend to lend credence to the Democrats' plan which is to stimulate the economy by injecting capital into the private sector rather than cutting taxes. The Republican mathematician may question the model as being flawed, and perhaps there is; and this "flaw" may be the restriction that the company *must follow* closely the optimum policy in order to increase its profits enough to cover the cost of the loan. A reasonable alternate policy (which gives higher profits to the firm) fails to achieve cost-coverage in many cases; would the government be wise enough to grease the right companies? Since it has no "self-interest", maybe not. The Democrat mathematician may respond that rather than a "flaw" the conclusion is the result of a *subtlety* in the model, a semantic difference – the company may have an *obligation* to follow the socially optimal policy (especially since it has taken the government money). After all, this is what happened after the banks, and other systemically important firms (for example, General Motors), received investments from the U.S. government during the 2008 Financial Crisis and the Great Recession.

where $X(t)$ is the fortune of the company at time t, W is a standard Brownian motion representing uncertainty and risk, and $Z(t)$ is the total profit (dividends) (which subtract from the fortune) taken up to time t. $Z(t)$ is assumed to be a nondecreasing, nonanticipating and càdlàg (right continuous with left limits) process. In particular, $Z(0)$ is allowed to be positive. Dutta & Radner (1999) assumed that the company chooses Z optimally to maximize the expected value of time-discounted profit without knowing the future of W, i.e., the market conditions. The company's *worth* is given by

$$\bar{V}(x) = \bar{V}(x;\mu,\sigma,r) = \sup_{\substack{Z:\, dZ/dt \geq 0, \\ \Delta Z(t) \leq X(t-)}} \left[Z(0) + \int_0^{\tau_0} e^{-rt} dZ(t) \right], \qquad (2)$$

where $r > 0$ is the discounting parameter and

$$\tau_0 = \tau_0^X = \inf\{t : X(t) \leq 0\},$$

is the time that the company goes bankrupt. For convenience, we will also use the notation

$$\int_{0-}^{\tau_0} e^{-rt} dZ(t) = Z(0) + \int_0^{\tau_0} e^{-rt} dZ(t).$$

The stipulation of Dutta & Radner (1999) that $\Delta Z(t) = Z(t) - Z(t-) \leq X(t-)$ means that the company cannot make a lump-sum dividend payment greater than its current fortune. The definition immediately implies that $\bar{V}(0) = 0$. Further $\bar{V}(x) \geq x$ since one strategy for taking profit is to take the fortune immediately ("take the money and run"); bankruptcy occurs at time $\tau_0 = 0+$. One notes that X depends upon Z and Z also depends upon X, which is somewhat paradoxical and requires an existence proof that there is a pair (X, Z) with this relationship, for every admissible strategy, Z, of profit taking. The standard way around this difficulty is to take the supremum over all Ω-space realizations of the pair of processes (X, Z).

A final point worth noting is that the Dutta–Radner choice of additive rather than multiplicative Brownian motion is an old and well-discussed issue, see Radner & Shepp (1996, p. 1390), where it is shown that inside a Samuelson–Black–Scholes multiplicative model the company either becomes infinitely rich ($\mu > r$) or $\bar{V}(x) \equiv x$ ("take the money and run"); i.e. it was shown that the multiplicative Brownian motion setup would not represent observed real-world behavior within the context of these types of models.

2.2. Solution to the Dutta–Radner model

As argued in Dutta & Radner (1999), when a dividend payment of amount dZ is made, the net change in the value function is $dZ(1 - \bar{V}'(x))$. Hence when $\bar{V}'(x) > 1$, dZ should be zero; when $\bar{V}'(x) \leq 1$, dZ should be as large as possible. Hence, if the value function $\bar{V}$ is concave, the optimal strategy, denoted by $\bar{Z}(t)$, should be a barrier payout process such that $d\bar{Z}/dt = 0$ or ∞ according as $X(t) < \bar{a}$ or $x > \bar{a}$ for some $\bar{a} \geq 0$. More precisely,

$$\bar{Z}(t) = \sup \{\max\{X(s) - \bar{a}, 0\} : \ 0 \leq s \leq t\}.$$

Assuming $\bar{V} \in C^2$, one may solve the Hamilton-Jacobi-Bellman equation

$$\max\{\mathcal{L}V(x), 1 - \bar{V}'(x)\} = 0, \qquad \mathcal{L} = -r + \mu^* \frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2},$$

with the initial condition $\bar{V}(0) = 0$ to obtain the solution

$$\begin{aligned} \bar{V}(x) &= \frac{e^{\gamma_+ x} - e^{\gamma_- x}}{\gamma_+ e^{\gamma_+ \bar{a}} - \gamma_- e^{\gamma_- \bar{a}}}, & 0 \leq x \leq \bar{a}, \\ \bar{V}(x) &= \frac{e^{\gamma_+ \bar{a}} - e^{\gamma_- \bar{a}}}{\gamma_+ e^{\gamma_+ \bar{a}} - \gamma_- e^{\gamma_- \bar{a}}} + (x - \bar{a}), & \bar{a} \leq x < \infty, \end{aligned} \tag{3}$$

where γ_+, γ_- are the roots of the indicial equation,

$$\gamma_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}.$$

The optimal threshold $\bar{a}$ can be most easily determined by the smooth-fit heuristic, $\bar{V}''(\bar{a}) = 0$, which yields

$$\bar{a} = \bar{a}(\mu, \sigma, r) = \begin{cases} \frac{1}{\gamma_+ - \gamma_-} \log\left(\frac{\gamma_-}{\gamma_+}\right)^2, & \text{if } \mu \geq 0, \\ 0, & \text{if } \mu < 0. \end{cases} \tag{4}$$

Using standard verification techniques, one can prove that the above solution is the true value function and the policy $\bar{Z}$ with reflection barrier $\bar{a}$ is indeed the optimal dividend payout policy.

The above result was first found by Dutta & Radner (1999)[3], who further showed that the company that follows the optimal policy will go bankrupt in a finite time with probability 1. The same model was considered at the same time by Radner & Shepp (1996), Jeanblanc-Picqué & Shiryaev (1995) and Asmussen & Taksar (1997), who also derived the optimal strategy of dividend payouts under various constraints on the payout rate. Both Dutta & Radner (1999) and Radner & Shepp (1996) also allow (μ, σ) to be part of the control, and Radner & Shepp (1996) found which specific value to use at any given value of $X(t) = x$, among $\{(\mu_i, \sigma_i) : i = 1, 2, \ldots, \}$. The solution gave rise to some

[3] According to Radner & Shepp (1996), the model was first proposed and solved by Dutta and Radner in an earlier manuscript in 1994.

surprising results, for example that if the company is nearly bankrupt then it should be very conservative and use the (μ_i, σ_i) pair with the smallest σ_i which seems to be paradoxical to many economists; see the work of Sheth et al. (2011). For simplicity in discussing the present question we will limit the company to only one corporate direction, i.e. $n = 1$.

Many variants of the Dutta–Radner model have also been proposed in the literature. For example, Choulli et al. (2003) assume that the control consists of two parts, one being the dividend payout process and the other affecting both the profit rate (drift) and the riskiness of the underlying fortune process. Décamps & Villeneuve (2007) extended the Dutta–Radner model to include a singular control process representing an investment. Recently, De Angelis & Ekström (2017) found the optimal policy for the finite horizon case. Other related works include Jgaard & Taksar (1999) and Asmussen et al. (2000).

2.3. Further insights into the Dutta–Radner model

From (4), we see that if $\mu < 0$, then $\bar{a} = 0$ and $\bar{V}(x, \mu, \sigma, r) \equiv x$, i.e. it is always optimal to take the money and run. However, if $\mu > 0$, the company is *profitable*, and then $\bar{a}(\mu, \sigma, r) > 0$; the threshold $\bar{a}$ goes to zero as $\mu \downarrow 0$ or as $\mu \uparrow \infty$ (see the left panel of Figure 1). Note that the optimal solution is of the form that the company takes profit only when it is "rich", i.e., when $X(t) \geq \bar{a}$. We also point out for later use that if $\mu \geq 0$,

$$\bar{V}(\bar{a}) = \frac{e^{\gamma_+ \bar{a}} - e^{\gamma_- \bar{a}}}{\gamma_+ e^{\gamma_+ \bar{a}} - \gamma_- e^{\gamma_- \bar{a}}} = \frac{\gamma_+ + \gamma_-}{\gamma_+ \gamma_-} = \frac{\mu}{r}. \tag{5}$$

We conclude this section by offering some further observations that, to the best of our knowledge, are not currently stated in the literature. When a sub-optimal reflecting barrier a is used, the formula (3) is still applicable with $\bar{a}$ replaced by a. That is, if we let

$$Z_a(t) = \sup \left\{ \max\{X(s) - a, 0\} : \; 0 \leq s \leq t \right\}, \tag{6}$$

denote a dividend payout policy which chooses to always pay out whatever amount that exceeds a constant threshold $a > 0$, then the expected time-discounted profit, denoted by $V(a)$, can be calculated as

$$V_a(x; \mu, \sigma, r) = E_x \int_{0-}^{\tau_0} e^{-rt} dZ_a(t) = \begin{cases} \dfrac{e^{\gamma_+ x} - e^{\gamma_- x}}{\gamma_+ e^{\gamma_+ a} - \gamma_- e^{\gamma_- a}}, & x \in [0, a], \\[2ex] V_a(a) + (x - a), & x > a. \end{cases} \tag{7}$$

To prove this, one only needs to verify that V_a satisfies $\mathcal{L}V(x) = 0$ for $x \in [0, a]$, $V_a(0) = 0$ and $V_a'(a) = 1$, and then may apply the argument we use in the

Step 1 of the verification proof provided in Section 6.1 of the present work (see also Remark 2.) The mapping $a \mapsto V_a(x)$ is always maximized at $a = \bar{a}$ for any $x \geq 0$ since $\bar{a}$ represents the optimal policy which maximizes the expected time-discounted profit (see the right panel of Figure 1 below).

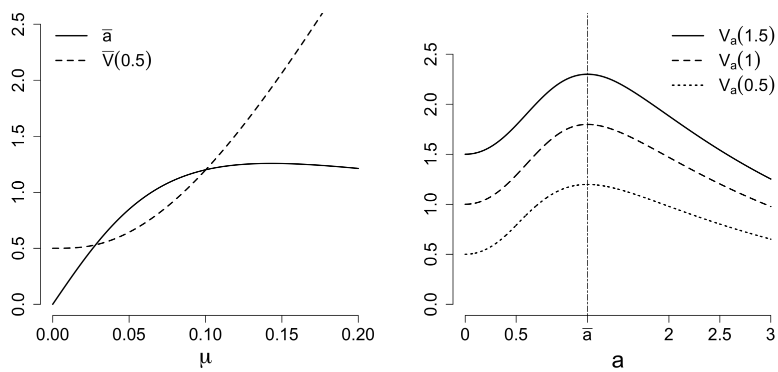

Figure 1: The Dutta–Radner model with $r = 0.05$ and $\sigma = 0.3$. The left panel plots the curves $\bar{a}(\mu, \sigma, r)$ and $\bar{V}(x; \mu, \sigma, r)$ with $x = 0.5$ and μ ranging from 0 to 3. Note that in this example $\bar{a}$ is maximized at $\mu = 0.143$. In the right panel, we fix $\mu = 0.1$ and plot the expected profit V_a against the reflecting barrier a. For any x, $V_a(x)$ is maximized at $a = \bar{a} = 1.200$.

3. Our new model: "greasing the wheels" with fiscal stimulus

The new model we shall consider generalizes that of Dutta & Radner (1999). The government will provide funds to the company through a direct loan, allowing it to receive up to a maximum rate, until bankruptcy. One might hope that there are some companies for which the expected additional profit from having the government funds is greater than the expected cost of the stimulus until bankruptcy. Indeed, we shall see in Section 4 that for *every* profitable company, the increased profit covers the loan, provided the company takes profits (dividends) in an optimal way to maximize its presumed objective. This is the main result of the present paper.

Following the notation of Dutta & Radner (1999), we will assume that $\mu > 0$, i.e., the company is profitable. To study the effect of fiscal stimulus, we shall consider the "greasing the wheels" argument of the Democrats. Suppose the government subsidizes the firm to put it in a better position and thereby increases the company's μ to μ^*. This might be done by a direct subsidy or by infrastructure improvements. The fiscal stimulus may be modelled as a government loan that injects cash into a given firm increasing their expected profit (μ to μ^*) that is to be paid back at interest rate r.[4] That is, the company may be supplied any amount less than or equal to $(\mu^* - \mu)dt$ in each interval dt, where $\mu^* \geq \mu$. The company may then choose an operating level at which to take profits. It may choose the Dutta–Radner optimal level $a^* = \bar{a}(\mu^*, \sigma, r)$ where $\bar{a}$ is as defined in (4) with μ^* in place of μ, but if it chooses this "greedy" or "socially-undesirable" strategy then the expected additional profit may not cover the expected loan cost. However, as we shall see, the company can do better, indeed so much better that it will *always* provide enough money to cover both the loan cost and provide the old profit. Henceforth we shall call a company "socially acceptable" if it meets the "socially acceptable" goal of covering the loan cost as well as, on average, providing the original profit.

To find a socially acceptable policy under borrowing from the government, suppose the company's worth at time t is again $X(t)$, which satisfies the stochastic differential equation in (1). Unlike in the Dutta–Radner formulation, we shall now allow *borrowing*; that is, $dZ/dt \geq -(\mu^* - \mu)$ instead of $dZ/dt \geq 0$. This means the firm can borrow at a limited rate, or, take negative dividends. As in Dutta & Radner (1999), we shall continue to require that $\Delta Z(t) = Z(t) - Z(t-) \leq X(t-)$ for any $t \geq 0$. The problem is now to maximize the mean discounted profit and to find the new value of the firm, defined as

$$V^*(x) = V^*(x, \mu, \mu^*, \sigma, r) = \sup_{\substack{Z:\, \Delta Z(t) \leq X(t-), \\ \frac{dZ}{dt} \geq -(\mu^* - \mu)}} E_x \int_{0-}^{\tau_0} e^{-rt} dZ(t), \tag{8}$$

where the sup is taken over all admissible choices of Z which are nonanticipating and cádlág with the derivative now allowed to be negative but which must be at least as large as $-(\mu^* - \mu)$. The negative increments of Z count negatively in computing the profit earned by the company. Note that any policy Z can be written in the following form

$$Z(t) = Z_+(t) - Z_-(t), \quad Z_+(0-) = Z_-(0-) = 0,$$

where $Z_+(t)$ represents the total dividends issued up to time t, which may be taxed, and $Z_-(t)$ is the total loans issued up to time t. Hence, we require $Z_\pm(t) \geq 0$ for any t and both Z_+ and Z_- are nondecreasing and nonanticipating processes. In addition, Z_- is assumed to be in C^1 with derivative

$$\frac{dZ_-(t)}{dt} \leq c = \mu^* - \mu.$$

[4] In a recent paper in the economics literature by Lucas (2016), it is also shown that government loans and subsidies to private firms can be an effective means of fiscal stimulus.

Let $\mathcal{A}$ denote the set of all possible processes (Z_+, Z_-) that satisfy the above conditions. We may now reformulate our stochastic control problem as

$$\begin{aligned} &dX(t) = \mu dt + \sigma dW(t) + dZ_-(t) - dZ_+(t), \quad 0 \le t \le \tau_0; \\ &X(0-) = x, \quad Z_-(0) = 0, \quad Z_+(0-) = 0; \\ &V^*(x, \mu, \mu^*, \sigma, r) = \sup_{(Z_+, Z_-) \in \mathcal{A}} E_x \int_{0-}^{\tau_0} e^{-rt} [dZ_+(t) - dZ_-(t)], \end{aligned} \tag{9}$$

where we recall bankruptcy occurs at τ_0.

The company will earn at least as much profit as in the Dutta–Radner case, since by choosing $Z_-(t) \equiv 0$, we obtain

$$\begin{aligned} V^*(x, \mu, \mu^*, \sigma, r) &\ge \sup_{(Z_+, Z_-) \in \mathcal{A},\, Z_- \equiv 0} E_x \int_{0-}^{\tau_0} e^{-rt} [dZ_+(t) - dZ_-(t)] \\ &= V^*(x, \mu, \mu, \sigma, r) = \bar{V}(x, \mu, \sigma, r). \end{aligned} \tag{10}$$

A similar argument reveals that, for any $\mu' \in [\mu, \mu^*)$, we have

$$V^*(x, \mu, \mu^*, \sigma, r) \ge V^*(x, \mu, \mu', \sigma, r).$$

We conclude this section with a brief discussion of our model. In some sense, we have chosen the model with a *built-in advantage* for the democratic side. On the other hand, society regularly objects to outsize profits taken by executives of money-losing companies. Perhaps a company that wants the loan (under which it makes more money) might be willing to agree to conditions on how and when it takes its profits. This appears credible and to justify the investment of the money provided. In Section 4, we will find the optimal strategy under the loan. We will show in Theorem 1 that the company should always take the loan *at the maximum rate*, and it should again issue profits at a threshold which is always lower than the threshold for the company with profit rate μ^* (see Proposition 2.)

4. Calculation of the optimal expected profit minus loan cost

In this section, we calculate the value function defined in (9). Just like in Dutta & Radner (1999), the optimal dividend payout control increases in local time, but now at a different threshold, $\hat{a}$, and under optimal control we show that the company continuously borrows at the maximum possible rate, $c = \mu^* - \mu$. Taking dividends in local time appears complicated but the thesis of Liu (2004) shows that if dividends are required to be lump-sum payments rather than "dribs and drabs," nearly the same total profit values are obtained.

We now state the main result of this paper.

Theorem 1 (verification)**.** *Let the function $\hat{V} \in C^2$ and $\hat{a} \geq 0$ be the solution to the free-boundary problem,*

$$\begin{cases} \mathcal{L}^*\hat{V}(x) = c, & x \in [0, \hat{a}], \\ \hat{V}(0) = 0, & \\ \hat{V}'(x) = 1, & x \in [\hat{a}, \infty), \\ \hat{V}''(x) = 0, & x \in [\hat{a}, \infty), \end{cases}$$

where $c = \mu^* - \mu \geq 0$ and

$$\mathcal{L}^* = -r + \mu^* \frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}. \tag{11}$$

Then $\hat{V}(x) = V^*(x)$, the solution to the new optimization problem given in (9)*. Let $\hat{Z} = (\hat{Z}_+, \hat{Z}_-)$ be the optimal policy and $\hat{X}$ be the corresponding fortune process. Then $\hat{Z}_-(t) = ct$ and $(\hat{X}, \hat{Z}_+)$ is the solution to the stochastic differential equation*

$$\hat{X}(t) = x + \mu^* t + \sigma W(t) - \hat{Z}_+(t), \qquad \hat{Z}_+(t) = \int_0^t I(\hat{X}(s) = \hat{a}) d\hat{Z}_+(s).$$

The proof is given in Section 6.1. The solution for $\hat{V}(x) = \hat{V}(x, \mu, \mu^*, \sigma, r)$ is

$$\hat{V}(x) = A_+(e^{\gamma_+^* x} - 1) + A_-(e^{\gamma_-^* x} - 1), \quad x \in [0, \hat{a}], \tag{12}$$

$$\hat{V}(x) = \hat{V}(\hat{a}) + (x - \hat{a}), \qquad x \in [\hat{a}, \infty), \tag{13}$$

where $\gamma_\pm^*$ now goes with μ^*, i.e.,

$$\gamma_\pm^* = \frac{-\mu^* \pm \sqrt{(\mu^*)^2 + 2r\sigma^2}}{\sigma^2},$$

quite analogous to $\gamma_\pm$ in the simpler Dutta–Radner problem, and constants

$$A_+ = \frac{-\gamma_-^* e^{-\gamma_+^* \hat{a}}}{\gamma_+^*(\gamma_+^* - \gamma_-^*)}, \qquad A_- = \frac{\gamma_+^* e^{-\gamma_-^* \hat{a}}}{\gamma_-^*(\gamma_+^* - \gamma_-^*)}.$$

The barrier $\hat{a}$ is now only *implicitly* defined as the root of the equation,

$$\frac{(-\gamma_-^*)e^{-\gamma_+^* a}}{(\gamma_+^*)(\gamma_+^* - \gamma_-^*)} - \frac{(\gamma_+^*)e^{-\gamma_-^* a}}{(-\gamma_-^*)(\gamma_+^* - \gamma_-^*)} - \frac{c}{r} = 0. \tag{14}$$

This optimal threshold, $\hat{a}$, in our new model does not have an analytical solution, unlike in the original model where it is known in closed form. The following proposition guarantees that, given $\mu > 0$, there always exists an appropriate solution with $\hat{a} > 0$ to the free-boundary problem in Theorem 1.

Proposition 1. *Assume $\mu > 0$ and $\mu^* \geq \mu$. The free-boundary problem in Theorem 1 has a unique solution $(\hat{V}, \hat{a})$ such that $\hat{V} \in C^2$ and $\hat{a} > 0$.*

Proof. We only need to verify that (14) has only one solution and it is positive. Denote the left-hand side of (14) by $f(a)$. It is straightforward to verify that f is monotone decreasing, $f(0) = \mu/r > 0$, and $f(\infty) = -\infty$. So there is one and only one root on $(0, \infty)$. □

From (10), we already know that $V^*(x, \mu, \mu^*, \sigma, r) \geq \bar{V}(x, \mu, \sigma, r)$; that is, by borrowing money from the government, the company is able to make a net profit at least as large as in the Dutta–Radner model without government loans. Now we prove that, as long as the optimal policy is used, the company can make a strictly larger expected net profit. Further, the more money a company can borrow from the government, the (strictly) larger expected net profit it can make.

Theorem 2. *Consider two pairs of drift parameters (μ_1, μ_1^*) and (μ_2, μ_2^*).*

(i) *If $\mu_1 = \mu_2 = \mu$ and $\mu_1^* > \mu_2^* \geq \mu$, then for any $x > 0$,*

$$V^*(x, \mu, \mu_1^*, \sigma, r) > V^*(x, \mu, \mu_2^*, \sigma, r).$$

(ii) *If $\mu_1^* = \mu_2^* = \mu^*$ and $\mu_2 < \mu_1 \leq \mu^*$, then for any $x > 0$,*

$$V^*(x, \mu_1, \mu^*, \sigma, r) > V^*(x, \mu_2, \mu^*, \sigma, r).$$

Proof. The proof for part (i) relies on the verification techniques used for proving Theorem 1 and requires the result of Proposition 2. Hence, the proof for this part is put in Section 6.2.

To prove part (ii), first notice that

$$V^*(x, \mu_2, \mu^*, \sigma, r) = E_x^2 \left[\int_{0-}^{\tau_0} e^{-rt} dZ_{\hat{a}_2}(t) - \int_{0-}^{\tau_0} (\mu^* - \mu_2) e^{-rt} dt \right],$$

where $\hat{a}_2 = \hat{a}(\mu_2, \mu^*, \sigma, r)$ and $Z_{\hat{a}_2}$ denotes a dividend payout policy with reflection barrier $\hat{a}_2$. Clearly, for another company characterized by the five-tuple $(x, \mu_1, \mu^*, \sigma, r)$, it can also use the policy $Z_{\hat{a}_2}$ to make dividend payments but borrow money at rate $\mu^* - \mu_1$. Hence,

$$V^*(x, \mu_1, \mu^*, \sigma, r) \geq E_x^1 \left[\int_{0-}^{\tau_0} e^{-rt} dZ_{\hat{a}_2}(t) - \int_{0-}^{\tau_0} (\mu^* - \mu_1) e^{-rt} dt \right],$$

and, when the policy on the right-hand side is used, $X(t)$ evolves according to

$$X(t) = x + \mu^* t + \sigma W(t) - Z_{\hat{a}_2}(t).$$

Therefore, the expectations E_x^1 and E_x^2 can be evaluated with respect to the same probability measure. Given any $x > 0$, we have $\tau_0 > 0$ under the above policy and thus the claim follows from the strict inequality $\mu^* - \mu_1 < \mu^* - \mu_2$. $\square$

We will compare the optimal threshold $\hat{a}$ with two other suboptimal choices:

$$\begin{aligned} \bar{a} &= \bar{a}(\mu, \sigma, r) = \hat{a}(\mu, \mu, \sigma, r), \\ a^* &= \bar{a}(\mu^*, \sigma, r) = \hat{a}(\mu^*, \mu^*, \sigma, r). \end{aligned} \tag{15}$$

The threshold $\bar{a}$ is defined in (4), which is the optimal threshold for the Dutt–Radner model where borrowing is not allowed. The threshold a^* represents a greedy strategy which only wants to maximize the dividend payouts and does not care about repaying the loan. We first prove that the new threshold $\hat{a}$ is always less than the old Dutta–Radner threshold $\bar{a}$. This is a second key result of our model. An immediate consequence is that since $\hat{a} \in (0, \bar{a})$, $\hat{a}$ can be computed numerically using a standard one-dimensional optimization algorithm.

Proposition 2. *Assume $\mu > 0$ and $\mu^* \geq \mu$. Then $\hat{a} = \hat{a}(\mu, \mu^*, \sigma, r) \leq \bar{a}(\mu, \sigma, r) = \bar{a}$ where $\bar{a}$ is the optimal threshold of the Dutta–Radner model. Further, if $\mu_1^* > \mu_2^* \geq \mu$, then we have the strict inequality*

$$\hat{a}(\mu, \mu_1^*, \sigma, r) < \hat{a}(\mu, \mu_2^*, \sigma, r).$$

Proof. Recall that in (5) we showed $\bar{V}(a) = \bar{V}(a) = \mu/r$. Now we note also that $V^*(\hat{a}) = \hat{V}(\hat{a}) = \mu/r$ due to (14). But for any x, $V^*(x) \geq \bar{V}(x)$, because V^* is the supremum over a larger class of admissible policies. Further, both $\bar{V}(x)$ and $V^*(x)$ are monotone increasing in x. Hence the fact $\bar{V}(a) = V^*(\hat{a})$ implies $\hat{a} \leq a$.

To prove the strict inequality, again let $f(a)$ denote the left-hand side of (14) and consider the mapping $\mu^* \mapsto f'(a; \mu^*)$ where f' denotes the derivative with respect to a. Routine but heavy calculation gives

$$\frac{\partial f'(a; \mu^*)}{\partial \mu^*} = -\frac{r\sigma^2 e^{-\gamma_+^* a}}{[(\mu^*)^2 + 2r\sigma^2]^{3/2}} \left\{ 1 + h(\mu^*) + e^{2h(\mu^*)}(h(\mu^*) - 1) \right\},$$

where $h(\mu^*) = a\sigma^{-2}\sqrt{(\mu^*)^2 + 2r\sigma^2}$. By computing the first and second derivatives, one can verify that $h \mapsto 1 + h + e^{2h}(h-1)$ is always positive on $(0, \infty)$. Hence, for any $a > 0$, $\mu^* \mapsto f'(a; \mu^*)$ is monotone decreasing. Recall that $f(0) = \mu/r > 0$ and f is monotone decreasing in a. Since for a larger μ^*, $f(a; \mu^*)$ decreases at a higher rate, we conclude that $\mu^* \mapsto \hat{a}(\mu, \mu^*, \sigma, r)$ is also monotone decreasing. $\square$

The greedy company only wants to maximize the dividend payout and does not care about repaying the loan. Then the problem reduces to the Dutta–

Radner model and the company should choose the threshold a^* defined in (15). We now prove in Proposition 3 below that a^* is always greater than $\hat{a}$ given $\mu^* > \mu > 0$. Hence, in the absence of being held accountable, the firm taking the loan will pay out dividends later. This is not surprising from a mathematical perspective. When we compare the Dutta–Radner model with drift μ^* with our new model with drift parameters μ and μ^*, the only difference is the objective function, and the additional loan cost in our model can be interpreted as a penalization on the "lifetime" of the company; without this penalty term, the firm wants to live longer and thus the dividend payout threshold is increased.

Proposition 3. *Let $a^* = \bar{a}(\mu^*, \sigma, r)$ be the optimal threshold of the Dutta–Radner model with profit rate μ^*. Then $a^* > \hat{a}(\mu, \mu^*, \sigma, r)$ if $\mu^* > \mu > 0$. More generally, if $\mu^* \geq \mu_1 > \mu_2$, then we have*

$$\hat{a}(\mu_1, \mu^*, \sigma, r) > \hat{a}(\mu_2, \mu^*, \sigma, r).$$

Proof. We only need to prove the general claim since $a^* = \hat{a}(\mu^*, \mu^*, \sigma, r)$ is a special case. Recall that, for any $\mu \leq \mu^*$, $\hat{a}(\mu, \mu^*, \sigma, r)$ is the solution to (14), and the left-hand side of (14), $f(a)$, is monotone decreasing in a. But observe that in (14), only $c = \mu^* - \mu$ depends on μ and

$$c_1 = \mu^* - \mu_1 < \mu^* - \mu_2 = c_2.$$

Since c only affects the vertical shift, but not the shape, of the function $f(a)$, we conclude that $\hat{a}(\mu_1, \mu^*, \sigma, r) > \hat{a}(\mu_2, \mu^*, \sigma, r)$. □

In our model, the net profit of a company can be written as the difference of two components:

$$E_x \int_{0-}^{\tau_0(a)} e^{-rt} dZ(t) = E_x \int_{0-}^{\tau_0(a)} e^{-rt} dZ_+(t) - E_x \int_{0-}^{\tau_0(a)} e^{-rt} dZ_-(t).$$

The first component is the positive dividend payouts to the firm, and the second is the cost of the loans (discounted). Now consider a policy denoted by Z_a^* which chooses to always borrow the money at the maximum rate c and pays out whatever amount that exceeds a constant threshold $a > 0$. Then the expected values of the time discounted total dividend payout and the cost of the loan can be written as

$$D_a(x; \mu^*, \sigma, r) = E_x \int_{0-}^{\tau_0(a)} e^{-rt} dL_a(t), \quad C_a(x; \mu, \mu^*, \sigma, r) = E_x \int_{0}^{\tau_0(a)} ce^{-rt} dt,$$

where $\tau_0(a)$ is the first time to hit zero of the process X_a, a Brownian motion with a reflecting barrier at a, and L_a is its local time at level a:

$$X_a(t) = x + \mu^* t + \sigma W(t) - L_a(t), \quad L_a(t) = \int_0^t I(X_a(s) = a) dL_a(s). \tag{16}$$

The process $L_a(t)$ is the same as $Z_a(t)$ defined in (6). Notice that D_a does not depend on μ and we already know how to compute D_a from Section 2:

$$D_a(x; \mu^*, \sigma, r) = V_a(x; \mu^*, \sigma, r),$$

where V_a is given in (7). Let $\hat{D} = D_{\hat{a}}$ and $\hat{C} = C_{\hat{a}}$ denote the above two components when the optimal policy $\hat{Z}$ is applied; thus $\hat{C} = \hat{V} - \hat{D}$. See the left panel of Figure 2 below for a numerical example.

The time-discounted mean cost of the loan with an arbitrary reflecting barrier a can be computed as

$$C_a(x) = \frac{c}{r} \left[1 - g(x)\right], \qquad g(x) = E_x \left[e^{-r\tau_0(a)}\right].$$

The calculation of the function g relies on the key observation that $g(X_a(t))e^{-rt}$ is a martingale (c.f. Darling & Siegert, 1953; Lehoczky, 1977; Ernst et al., 2018). By Itô calculus, the martingale property implies that $g'(a) = 0$ and $\mathcal{L}^* g(x) = 0$ for $x \in [0, a]$ where the operator $\mathcal{L}^*$ is defined in (11). Another boundary condition is given by the fact that $g(0) = 1$. The solution is given in the following proposition.

Proposition 4. *Let $\tau_0(a)$ be the first time to hit zero of the process $X_a(t)$ defined in* (16)*. Then,*

$$g(x; \mu, \mu^*, \sigma, r) = E_x\, e^{-r\tau_0(a)} = \frac{\gamma_-^* e^{-\gamma_+^*(a-x)} - \gamma_+^* e^{-\gamma_-^*(a-x)}}{\gamma_-^* e^{-\gamma_+^* a} - \gamma_+^* e^{-\gamma_-^* a}},$$

which is the solution to the differential equation $\mathcal{L}^* g(x) = 0$ with boundary conditions $g'(a) = 0$ and $g(0) = 1$.

Proof. Using Itô calculus and the fact that $\mathcal{L}^* g(x) = 0$ and $g'(a) = 0$, we have

$$de^{-rt} g(X_a(t)) = \sigma e^{-rt} g'(X_a(t)) dW_t.$$

Integrating from 0 to $\tau_0(a)$, taking expectation and using $g(0) = 1$, we obtain

$$E_x e^{-r\tau_0(a)} - g(x) = E_x \int_0^{\tau_0(a)} \sigma e^{-rt} g'(X_a(t)) dW_t.$$

For any $t \in [0, \tau_0(a)]$, clearly $g'(X_a(t))$ is bounded since $X_a(t) \in [0, a]$. Hence, the integral on the right-hand side is zero, which completes the proof. □

Now for a strategy Z_a^* with reflecting barrier $a > 0$, the expected net profit can be computed as

$$\begin{aligned} V_a^*(x;\mu,\mu^*,\sigma,r) &= D_a(x;\mu^*,\sigma,r) - C_a(x;\mu,\mu^*,\sigma,r) \\ &= V_a(x;\mu^*,\sigma,r) - \frac{c}{r}[1 - g(x;\mu,\mu^*,\sigma,r)]. \end{aligned} \tag{17}$$

In particular, for a greedy firm which uses the threshold a^*, we also have $D_a(x;\mu^*,\sigma,r) = \bar{V}(x;\mu^*,\sigma,r)$. The mapping $a \mapsto V_a^*(x)$ is maximized at $a = \hat{a}$. The numerical example provided in the right panel of Figure 2 illustrates how the three functions V_a^*, D_a, C_a change with a.

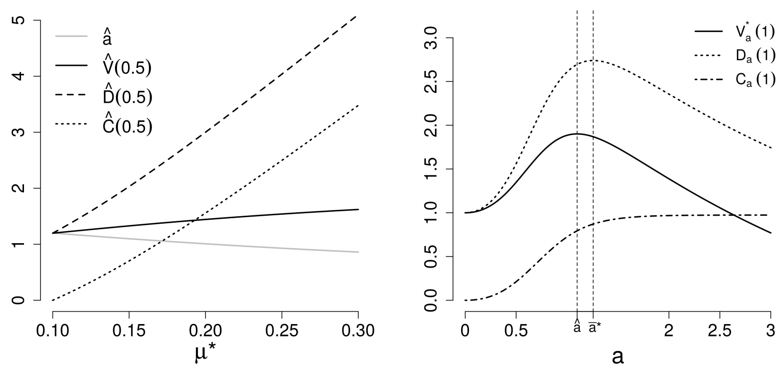

Figure 2: Our new model with $\mu = 0.1$, $r = 0.05$ and $\sigma = 0.3$. The left panel plots the curves $\hat{a}(\mu,\mu^*,\sigma,r)$, $\hat{V}(x;\mu,\mu^*,\sigma,r)$, $\hat{D}(x;\mu^*,\sigma,r)$ and $\hat{C}(x;\mu,\mu^*,\sigma,r)$ with $x = 0.5$ and μ^* ranging from 0.1 to 0.3. In the right panel, we fix $\mu^* = 0.15$, $x = 1$ and plot the expected net profit V_a^*, the expected dividend payout D_a, and the expected loan cost C_a against the reflecting barrier a. $V_a^*(x)$ is maximized at $a = \hat{a} = 1.099$ and D_a is maximized at $a = a^* = 1.257$.

5. Discussion

The Democratic policy of greasing the wheels through fiscal stimulus promises to bring benefit to all. Indeed, if it were true that government loans to some profitable firms would increase the rate of income to the point that the increased profit would pay for itself in expectation then, as the Democrats have been arguing, this extra money could be used to supplement taxes and the extra capital above the cost of the loans could be used in many ways, including creating jobs. Indeed, our new model in Section 4 confirms this hypothesis, *provided* that loans are made to profitable companies who follow the socially acceptable policy.

Unfortunately, if the company uses the Dutta–Radner policy with μ^* in place of μ, this will lead to choosing a dividend threshold which is not $\hat{a}$, the one for the socially acceptable policy of maximizing $\hat{V}(x)$. The expected net profit of this company would be $V^*_{a^*}$, which can be computed using (17). If $\mu^* > \mu$, the expected total dividend payout, $D_a(x;\mu^*,\sigma,r) = \bar{V}(x;\mu^*,\sigma,r)$, is always greater than $\bar{V}(x;\mu,\sigma,r)$, the maximum total dividend payout without borrowing money, which is the primary motivation behind such a greedy strategy. To better understand the consequences of this strategy, we herein investigate the following two questions.

(i) Is it true that the expected net profit, $V^*_{a^*}(x;\mu,\mu^*,\sigma,r)$, is greater than or equal to $\bar{V}(x;\mu,\sigma,r)$? If not, it means that the company would be foolish to be greedy, since it is likely to obtain a higher expected net profit by not borrowing any money from the government.

(ii) Is it true that the expected total dividend payout, $\bar{V}(x;\mu^*,\sigma,r)$, is greater than or equal to the expected loan cost, $C_a(x;\mu,\mu^*,\sigma,r)$? If not, this implies that it could be dangerous of the government to lend money to some firms since they may not even be able to repay the loan (in expectation).

We performed various numerical experiments and found that the answer to both questions to be "not necessarily." For the first question, we need μ to be relatively large compared to r for the statement to be true. In Figure 3, we give a numerical example of two different companies. The interest rate is $r = 0.05$; the profit rate $\mu = 0.08$ for Company 1 and $\mu = 0.06$ for Company 2. For Company 1, $V^*_{a^*}(x;\mu,\mu^*,\sigma,r)$ keeps increasing as μ^* increases, which means the the more money it can borrow, the larger net profit it can make by using the greedy reflecting barrier a^*. However, for Company 2, the curve has a different shape. Indeed, when no money is borrowed, Company 2 has $\bar{V}(1) = 1.24$. But if $\mu^* = 0.16$ (which corresponds to the borrowing rate $c = 0.1$) and it uses a^*, then $V^*_{a^*}(1)$ drops to 1.17.

For the second question, numerics show that the firms with greedy policies are usually able to repay the loan unless both μ and x are very small. For a concrete example, let $\mu = 0.055, \sigma = 0.1, r = 0.05$ and $x = 0.05$. Numerics show that when $\mu^* > 0.34$, the expected total dividend payout of a greedy firm cannot cover the expected loan cost and thus the net profit would be negative. Although such bad cases are very rare, it may be better if the government can step in to ensure that the "correct" dividend policy is utilized, or the government might try to recognize which companies would be in the "good" class where the increased profits will pay for the loan. We leave the question of how to tell which companies are good in this sense to future research.

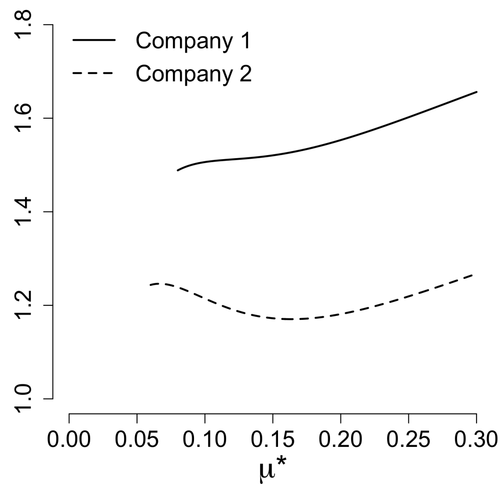

Figure 3: Behavior of the "greedy" companies with $r = 0.05$ and $x = 1$. We plot the net profit when a greedy policy is used, $V^*_{a^*}(x; \mu, \mu^*, \sigma, r)$, against the parameter μ^*. The first company is characterized by $\mu = 0.08$ and $\sigma = 0.3$, while the second has $\mu = 0.06$ and $\sigma = 0.3$.

6. Proofs

6.1. Proof for Theorem 1

The verification consists of two steps. First, we need to check that $\hat{V}(x)$ is indeed the expected value of the time-discounted profit when we apply the optimal control function $(\hat{Z}_+, \hat{Z}_-)$. Second, we need to prove that no other policy can do better. The latter requires the following lemma.

Lemma 1. *The solution $\hat{V}$ given in* (12) *and* (13) *satisfies $\hat{V}'(x) \geq 1$ and $\mathcal{L}^*\hat{V}(x) \leq c$ for any $x \geq 0$. Consequently, the following Hamilton-Jacobi-Bellman equation holds:*

$$\max\{\mathcal{L}^*\hat{V}(x) - c, 1 - \hat{V}(x)\} = 0, \qquad \forall x \geq 0.$$

Proof. We first show that $\hat{V}''(x) \leq 0$ for any $x \geq 0$. By the expression of $\hat{V}$ given in (12) and (13), this is equivalent to proving that, for any $x \in [0, \hat{a}]$,

$$A_+(\gamma^*_+)^2 e^{(\gamma^*_+ - \gamma^*_-)x} \leq -A_-(\gamma^*_-)^2. \tag{18}$$

Since by definition $\gamma^*_+ > 0$ and $\gamma^*_- < 0$, we only need show (18) holds true for $x = \hat{a}$. But we already know that $V''(\hat{a}) = 0$ and thus $A_+(\gamma^*_+)^2 e^{(\gamma^*_+ - \gamma^*_-)\hat{a}} = -A_-(\gamma^*_-)^2$. Hence we conclude $\hat{V}''(x) \leq 0$.

Since $\hat{V}'(\hat{a}) = 1$ for any $x \geq \hat{a}$, the non-positivity of $\hat{V}''$ implies that $\hat{V}'(x) \geq 1$ for any $x \geq 0$. To prove $\mathcal{L}^*\hat{V}(x) \leq c$, notice that $\hat{V}(x) > \hat{V}(\hat{a})$, $\hat{V}'(x) = \hat{V}'(\hat{a})$, $\hat{V}''(x) = \hat{V}''(\hat{a})$ for any $x > \hat{a}$. The claim then follows from the condition that $\mathcal{L}^*\hat{V}(\hat{a}) = c$. □

Remark 1. The smooth-fit condition, $\hat{V}''(\hat{a}) = 0$, is critical in the above proof. Assume all the other conditions in Theorem 1 are satisfied. Now if $\hat{V}''(\hat{a}-) > 0$, then $\hat{V}'(\hat{a} - \epsilon) < 1$ for some $\epsilon > 0$ since $\hat{V}'(\hat{a}) = 1$. If $\hat{V}''(\hat{a}-) < 0$, one can show that $\mathcal{L}^*\hat{V}(\hat{a}+) > c$, since there is a jump increase in $\hat{V}''(x)$ at $x = \hat{a}$ and $\mathcal{L}^*\hat{V}(\hat{a}-) = c$. Therefore, the smooth-fit condition is necessary for the Hamilton-Jacobi-Bellman equation to hold true.

Now we present our verification proof.

Step 1. Consider the case $x \in [0, \hat{a}]$. The process $\hat{Z}_+(t)$ is the local time of the process $\hat{X}$ at the level $\hat{a}$ and thus it is continuous, nondecreasing, nonanticipating and adapted. Note that such a pair $(\hat{X}, \hat{Z}_+)$ always exists (Karatzas & Shreve, 1998, Chap. 3.6).

Define $\alpha_t = \tau_0^{\hat{X}} \wedge t$ and consider the process $e^{-r\alpha_t}\hat{V}(\hat{X}(\alpha_t))$. By Itô's formula, we have

$$\begin{aligned} & e^{-r\alpha_t}\hat{V}(\hat{X}(\alpha_t)) \\ = {} & \hat{V}(x) + \int_0^{\alpha_t} e^{-rs}\mathcal{L}^*\hat{V}(\hat{X}(s))ds + \int_0^{\alpha_t} e^{-rs}\hat{V}'(\hat{X}(s))\left[\sigma dW(s) - d\hat{Z}_+(s)\right]. \end{aligned}$$

For any $s \in [0, \tau_0^{\hat{X}}]$, we always have $\hat{X}(s) \in [0, \hat{a}]$ and thus $\mathcal{L}^*\hat{V}(\hat{X}(s)) = c$ and $\hat{V}'(\hat{X}(s))$ stay bounded. Hence, the integral with respect to dW is zero. Taking expectation on both sides we obtain

$$E_x\, e^{-r\alpha_t}\hat{V}(\hat{X}(\alpha_t)) = \hat{V}(x) + c\int_0^{\alpha_t} e^{-rs}ds - E_x \int_0^{\alpha_t} e^{-rs}d\hat{Z}_+(s), \tag{19}$$

where we have also used the fact that $d\hat{Z}_+(t) = 0$ if $\hat{X}(t) \neq \hat{a}$ and $\hat{V}'(\hat{a}) = 1$. Next we claim that

$$\lim_{t\to\infty} E_x\, e^{-r\alpha_t}\hat{V}(\hat{X}(\alpha_t)) = 0. \tag{20}$$

If the bankruptcy happens, i.e. $\tau_0^{\hat{X}} < \infty$, then $\hat{V}(\hat{X}(t)) = 0$ for any $t \geq \tau_0^{\hat{X}}$; if $\tau_0^{\hat{X}} = \infty$, notice that $\hat{V}(\hat{X}(t))$ always stays bounded and $e^{-r\alpha_t} \to 0$. Hence, we obtain (20). Letting $t \to \infty$ in (19) and applying the monotone convergence theorem to the integral with respect to $d\hat{Z}_+$, we obtain that

$$\hat{V}(x) = E_x \int_{0-}^{\tau_0^X} e^{-rt}\left[d\hat{Z}_+(t) - d\hat{Z}_-(t)\right]. \tag{21}$$

In particular, we have established the equality (21) for $x = \hat{a}$, which, together with the expression of $\hat{V}$ given in (13), can be used to easily show that (21) also holds for $x \in (\hat{a}, \infty)$.

Remark 2. The smooth-fit condition, $\hat{V}''(\hat{a}) = 0$, is not used in Step 1. Now one can easily verify that the function V_a^* given in (17) is the solution to the ordinary differential equation

$$\mathcal{L}^* v(x) = c, \quad x \in [0, a],$$

with boundary conditions $v(0) = 0, v'(a) = 1$. Hence, the above argument also proves that V_a^* is indeed the expected net profit of the policy Z_a^* which pays the dividend using policy $Z_a(t)$ defined in (6) and borrows money at a constant rate c.

Step 2. For any admissible policy $Z = (Z_+, Z_-)$, let

$$V(x; Z) = E_x \int_{0-}^{\tau_0(X)} e^{-rt} \left[dZ_+(t) - dZ_-(t) \right]$$

be the expected value of the time-discounted profit. We need to show that $V(x; Z) \le \hat{V}(x)$. By assumption, Z_- is a continuous process but Z_+ is not necessarily. So we let Z_+^c be the continuous part of Z_+. The dynamics of the process X are given by (9). We apply Itô's formula to obtain

$$\hat{V}(x) = -\int_0^{t \wedge \tau_0^X} \sigma e^{-rs} \hat{V}'(X(s)) dW(s) + I_1 + I_2 - I_3 + I_4 \tag{22}$$

where

$$\begin{aligned} I_1 &= \int_0^{t \wedge \tau_0^X} e^{-rs} \left[\hat{V}'(X(s))(-dZ_-(s) + c\, ds) - \mathcal{L}^* \hat{V}(X(s)) ds \right], \\ I_2 &= \int_0^{t \wedge \tau_0^X} e^{-rs} \hat{V}'(X(s)) dZ_+^c(s), \\ I_3 &= \sum_{0 \le s \le (t \wedge \tau_0^X)} e^{-rs} \left\{ \hat{V}(X(s)) - \hat{V}(X(s-)) \right\}, \\ I_4 &= e^{-r(\tau_0^X \wedge t)} \hat{V}(X(t)). \end{aligned} \tag{23}$$

From Lemma 1, we have $\hat{V}'(x) \ge 1$ and $\mathcal{L}^* \hat{V}(x) \le c$. Moreover, by assumption $-dZ_-(s) + c\, ds \ge 0$. Therefore, we obtain

$$I_1 \ge \int_0^{t \wedge \tau_0^X} e^{-rs} \left[-dZ_-(s) + c\, ds - c\, ds \right] = -\int_0^{t \wedge \tau_0^X} e^{-rs} dZ_-(s)$$

and

$$I_2 \geq \int_0^{t\wedge\tau_0^X} e^{-rs} dZ_+^c(s).$$

Further, $\hat{V}'(x) \geq 1$ also implies that

$$I_3 \leq -\sum_{0\leq s\leq (t\wedge\tau_0^X)} e^{-rs}\Delta Z_+(s),$$

where we have used the fact that $X(s) - X(s-) = -(Z_+(s) - Z_+(s-))$. Taking expectation on both sides of (22) and using the boundedness of $\hat{V}'$, we get

$$\hat{V}(x) \geq E_x \int_{0-}^{t\wedge\tau_0^X} e^{-rs}\left[dZ_+(s) - dZ_-(s)\right],$$

since clearly $I_4 \geq 0$. Finally by letting $t \to \infty$, we obtain the inequality $V(x;Z) \leq \hat{V}(x)$, which completes the proof.

6.2. Proof for Theorem 2 (i)

The value function $V^*(x,\mu,\mu_2^*,\sigma,r)$ is attained by a policy where money is borrowed at rate $\mu_2^* - \mu$ and dividend payments are made at a barrier

$$b = \hat{a}(\mu,\mu_2^*,\sigma,r).$$

We let Z_b denote this dividend payout policy which is defined in (6), and use

$$\hat{Z}_2(t) = Z_b(t) - (\mu_2^* - \mu)t,$$

to denote the full control function. Clearly $\hat{Z}_2(t)$ is also admissible to a company characterized by (x,μ,μ_1^*,σ,r) since $\mu_1^* > \mu_2^*$, and we want to show that for any $x > 0$

$$\hat{V}(x,\mu,\mu_1^*,\sigma,r) > V(x,\mu,\mu_1^*,\sigma,r;\, \hat{Z}_2).$$

Recall that by (22) in the verification proof, we have

$$\hat{V}(x,\mu,\mu_1^*,\sigma,r) \geq E_x\, \left(I_1 + I_2 - I_3\right),$$

where I_1, I_2, I_3 are defined in (23). Now all we need to show is that, when $Z = \hat{Z}_2$, we have the strict inequality

$$I_1 + I_2 - I_3 > \int_{0-}^{t\wedge\tau_0^X} e^{-rs} d\hat{Z}_2(s) = \int_{0-}^{t\wedge\tau_0^X} e^{-rs}\left[dZ_b(s) - (\mu_2^* - \mu)ds\right].$$

Recall that $\hat{V}(x) = \hat{V}(x; \mu, \mu_1^*, \sigma, r)$ is the solution to the free-boundary problem described in Theorem 1 and the boundary solution is given by $\hat{a} = \hat{a}(\mu, \mu_1^*, \sigma, r)$, which is smaller than b by Proposition 2. Using Lemma 1, one can verify that

$$\begin{aligned} \mathcal{L}_1^* \hat{V}(x; \mu, \mu_1^*, \sigma, r) &< c, \quad x \in (\hat{a}, b], \\ \hat{V}'(x; \mu, \mu_1^*, \sigma, r) &> 1, \quad x \in (0, \hat{a}), \\ \hat{V}'(x; \mu, \mu_1^*, \sigma, r) &= 1, \quad x = b, \end{aligned}$$

where

$$\mathcal{L}_1^* = -r + \mu_1^* \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2}.$$

Further, when $Z = \hat{Z}_2$, we have

$$-dZ_-(s) + cds = -(\mu_2^* - \mu)ds + (\mu_1^* - \mu)ds = (\mu_1^* - \mu_2^*)ds > 0,$$

which implies that for any $x \in (0, \hat{a}) \cup (\hat{a}, b]$,

$$\hat{V}'(x; \mu, \mu_1^*, \sigma, r)(-dZ_-(s) + c\, ds) - \mathcal{L}_1^* \hat{V}(x)ds > -dZ_-(s) = -(\mu_2^* - \mu)ds.$$

The Lebesgue measure of the set $\{0 \le s \le \tau_0^X : X(s) = \hat{a}\}$ is zero. Hence, when $Z = \hat{Z}_2$ is applied and $X(t)$ starts from $x > 0$, we have, with probability 1,

$$\begin{aligned} I_1 &= \int_0^{t \wedge \tau_0^X} e^{-rs} \left[\hat{V}'(X(s))(-dZ_-(s) + c\, ds) - \mathcal{L}^* \hat{V}(X(s))ds \right] \mathbb{1}_{\{X(s) \neq \hat{a}\}} \\ &> \int_{0-}^{t \wedge \tau_0^X} -e^{-rs}(\mu_2^* - \mu)ds. \end{aligned}$$

Note that we have the equality

$$I_2 - I_3 = \int_{0-}^{t \wedge \tau_0^X} e^{-rs} dZ_b(s).$$

By the same arguments used in the verification proof, we conclude that

$$\hat{V}(x, \mu, \mu_1^*, \sigma, r) \ge E_x(I_1 + I_2 - I_3) > E_x \int_{0-}^{t \wedge \tau_0^X} e^{-rs} d\hat{Z}_2(s) = \hat{V}(x, \mu, \mu_2^*, \sigma, r).$$

Acknowledgments

Philip Ernst is grateful to ARO-YIP-71636-MA, NSF DMS-1811936, and ONR N00014-18-1-2192 for their support of this research.

References

Asmussen, S., Højgaard, B., & Taksar, M. (2000). Optimal risk control and dividend distribution policies. example of excess-of loss reinsurance for an insurance corporation. *Finance and Stochastics*, *4*, 299–324.

Asmussen, S., & Taksar, M. (1997). Controlled diffusion models for optimal dividend payout. *Insurance: Mathematics and Economics*, *20*, 1–15.

Choulli, T., Taksar, M., & Zhou, X. Y. (2003). A diffusion model for optimal dividend distribution for a company with constraints on risk control. *SIAM Journal of Control and Optimization*, *41*, 1946–1979.

Darling, D. A., & Siegert, A. (1953). The first passage problem for a continuous markov process. *The Annals of Mathematical Statistics*, (pp. 624–639).

De Angelis, T., & Ekström, E. (2017). The dividend problem with a finite horizon. *The Annals of Applied Probability*, *27*, 3525–3546.

Décamps, J.-P., & Villeneuve, S. (2007). Optimal dividend policy and growth option. *Finance and Stochastics*, *11*, 3–27.

Dutta, P. K., & Radner, R. (1999). Profit maximization and the market selection hypothesis. *The Review of Economic Studies*, *66*, 769–798.

Ernst, P., Rogers, L. C. G., & Zhou, Q. (2018). When is it best to follow the leader? *arXiv preprint arXiv:1804.05479*, .

Jeanblanc-Picqué, M., & Shiryaev, A. N. (1995). Optimization of the flow of dividends. *Russian Mathematical Surveys*, *50*, 257.

Jgaard, B. H., & Taksar, M. (1999). Controlling risk exposure and dividends payout schemes: insurance company example. *Mathematical Finance*, *9*, 153–182.

Karatzas, I., & Shreve, S. E. (1998). Brownian motion. In *Brownian Motion and Stochastic Calculus* (pp. 47–127). Springer.

Lehoczky, J. P. (1977). Formulas for stopped diffusion processes with stopping times based on the maximum. *The Annals of Probability*, *5*, 601–607.

Liu, J. (2004). *Stochastic Control Problems with Limitations on the Set of Allowable Controls*. Ph.D. thesis Rutgers.

Lucas, D. (2016). Credit policy as fiscal policy. *Brookings Papers on Economic Activity*, *2016*, 1–57.

Radner, R., & Shepp, L. (1996). Risk vs. profit potential: A model for corporate strategy. *Journal of Economic Dynamics and Control*, *20*, 1373–1393.

Sheth, A., Shepp, L., & Palmon, O. (2011). Risk-taking, financial distress and innovation. *Academy of Business Journal: Special Issue on the Global Debt Crisis*, *2*, 5–18.